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Gravity-capillary waves in the presence of constant vorticity

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Abstract – Periodic and solitary gravity-capillary waves propagating at a constant velocity at the surface of a fluid of finite depth are considered. The vorticity in the fluid is assumed to be constant. Analytical solutions are presented for waves of small amplitude. For waves of large amplitude, numerical solutions are computed by boundary integral equation methods. The results unify previous findings for irrotational gravity capillary waves and gravity waves with constant vorticity. In particular solitary waves with oscillatory tails and branches of solutions which exist only for waves of large amplitude are found. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

In recent years new properties of travelling waves have been discovered. Firstly it was shown that irrotational gravity capillary solitary waves are usually characterized by oscillatory tails in the far field. Secondly new families of pure gravity waves (i.e. waves in the absence of surface tension) were found in the presence of constant vorticity. These waves exist only for sufficiently large amplitude. The purpose of this paper is to develop a theory of gravity capillary waves in the presence of constant vorticity. We show that there are many different branches of solutions and we identify for which ranges of values of the parameters these solutions occur. Our findings are based on analytical solutions for waves of small amplitude and numerical solutions for waves of large amplitude.

Gravity-capillary solitary waves were considered by Korteweg and de Vries [1] in their classical 1895 paper. By using a formal perturbation expansion, they derived an approximate differential equation for the unknown shape of the free surface. This equation is known as the Korteweg–de Vries equation: it predicts elevation solitary waves for $0 \leqslant \tau < 1/3$ and depression waves for $\tau > 1/3$. Here

$$\tau = \frac{T}{\rho g H^2} \tag{1}$$

is the Bond number, g is the acceleration of gravity, T is the surface tension, ρ is the density of the fluid and H is the depth. Recent numerical and analytical calculations have shown that the predictions of Korteweg and de Vries are correct for $\tau > 1/3$ (see for example Hunter and Vanden-Broeck [2] for numerical evidence). However for $\tau < 1/3$, there is a train of ripples on the tail of the solitary waves which is not predicted by the Korteweg–de Vries equation (Hunter and Vanden-Broeck [2]; Iooss and Kirchgassner [3]; Sun [4]; Beale [5] and others). The solitary waves with oscillatory tails are often referred to as generalized solitary waves to distinguish them from true solitary waves which are characterized by a flat free surface in the far field.

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The above results are weakly nonlinear results for solitary waves of small amplitude. Hunter and Vanden-Broeck [2] and Vanden-Broeck [6] extended these results numerically to nonlinear waves of arbitrary amplitude.

Benjamin [7] derived asymptotic solutions for gravity solitary waves of small amplitude in the presence of vorticity. For constant vorticity his results show that for each value of vorticity, there is a one parameter family of solutions which bifurcates from the uniform shear flow.

Teles da Silva and Peregrine [8], Pullin and Grimshaw [9] and Vanden-Broeck [10,11] calculated numerically gravity solitary waves with constant vorticity. Their results extend Benjamin's one parameter family of solutions for waves of finite amplitude. Furthermore Vanden-Broeck [10] showed that there are in addition branches of solutions which do not bifurcate from a uniform shear current. Some of these branches are characterized by a limiting configuration with a 120° angle at the crest of the wave. Others have a limiting configuration consisting of a circular region of fluid in rigid body rotation.

Vanden-Broeck [11,12] showed that there are also families of solutions which approach a limiting configuration consisting of a wave with an arbitrary number of circular regions of fluid in rigid body rotation on top of its crest. These solutions were found by a continuation method starting with a solution close to the limiting configuration with a 120° angle at the crest of the wave.

The problem for gravity-capillary waves in the presence of constant vorticity is formulated in section 2. Analytical solutions are presented in sections 3 and 4. In section 3, we derive a linear theory for periodic waves under the assumption that the amplitude of the waves is small. In section 4, we study long waves of small amplitude and derive a Korteweg–de Vries equation. Solitary wave solutions are discussed. The analysis generalizes the work of Benjamin [7] for pure gravity waves in the presence of vorticity.

In section 5, we use the analytical results of sections 3 and 4 to identify ranges of values of the parameters where we can expect the fully nonlinear calculations to yield solitary waves with oscillatory tails of constant amplitude.

In section 6, we use numerical schemes to compute periodic and solitary gravity capillary waves in the presence of constant vorticity. The schemes are based on the reformulation of the problem as integro-differential equations (see section 2) and combine ideas from Hunter and Vanden-Broeck [2], Simmen and Saffman [13], Teles Da Silva and Peregrine [8] and Vanden-Broeck [6,10–12]. Typical numerical results are presented and discussed.

2. Formulation

In this section we formulate the problem of gravity capillary waves in the presence of constant vorticity as integro-differential equations. These equations will be solved numerically in section 6. The basic parameters and boundary conditions introduced in this section will also be used in the analytical calculations of sections 3–5. We first consider solitary waves in subsection 2.1. The appropriate modifications to formulate the problem of periodic waves are then described in subsection 2.2. These formulations generalize those in Vanden-Broeck [10, 12] for irrotational waves.

2.1. Solitary waves

We consider a two-dimensional solitary wave in a inviscid incompressible fluid which is bounded below by a horizontal bottom and above by a free surface.

Gravity and surface tension are taken into account. The flow is assumed to be rotational with a constant vorticity Ω. We take a frame of reference (x, y) in which the flow is steady. We choose the origin so that the x-axis is along the horizontal bottom, and the y-axis is a line of symmetry of the wave. As we shall see the problem is characterized by the Bond number τ (see (1)) and the dimensionless numbers

$$\omega = \frac{\Omega H}{C}, \tag{2}$$

$$F^2 = \frac{C^2}{gH}, \tag{3}$$

$$\alpha = \frac{A}{H}. \tag{4}$$

Here H the depth of the fluid at infinity, C the value of the velocity on the free surface at infinity, g the acceleration due to gravity and A the elevation of the crest of the wave on top of the level of the free surface at infinity. The parameter F is the Froude number. We introduce dimensionless variables by choosing H as the unit length and C as the unit velocity. The flow is described by a stream function $\Psi(x, y)$ satisfying

$$\nabla^2 \Psi = -\omega \tag{5}$$

in the flow domain.

We write the stream function Ψ as the sum of a function ψ and a particular solution of (5):

$$\Psi = \psi - \frac{\omega}{2} y^2 + (1 + \omega) y. \tag{6}$$

Substituting (6) into (5) we obtain $\nabla^2 \psi(x, y) = 0$.

Next we define the quantity $W(z) = u - iv = \psi_y + i\psi_x$. It is an analytic function of $z = x + iy$. The fluid velocity vector $(\Psi_y, -\Psi_x)$ is then defined in terms of u and v by $(u - \omega y + \omega + 1, v)$. In order to satisfy the kinematic condition $v = 0$ on the bottom we reflect the flow in the bottom.

We apply Cauchy integral formula to the function $W(z)$ on a contour consisting of the free surface, its image into the bottom and two vertical lines at $x = \pm\infty$. Since $W(z)$ vanishes at infinity, there are no contributions from the lines at $x = \pm\infty$ and we have

$$W(z) = -\frac{1}{\pi i} \oint_{C_*} \frac{W(\zeta)}{\zeta - z} \, d\zeta, \tag{7}$$

where z is a point on the free surface. The contour C_* consists of the free surface and its image into the bottom. The integral in (7) is a Cauchy principal value.

We parameterize the free surface by $x = X(t)$, $y = Y(t)$ where t is the arclength. We choose $t = 0$ at the crest of the wave. Then

$$X'(t)^2 + Y'(t)^2 = 1, \tag{8}$$

$$X(0) = 0, \qquad Y(0) = 1 + \alpha, \tag{9}$$

where α is the amplitude parameter defined in (4). We consider u and v to be functions of t. Taking the real part of (7) and using the symmetry of the wave with respect to the y-axis, we obtain, after some algebra

$$
\begin{aligned}
&\pi u(t) \\
&\quad = \int_0^\infty \frac{(X(s)-X(t))(u(s)Y'(s)-v(s)X'(s))-(Y(s)-Y(t))(u(s)X'(s)+v(s)Y'(s))}{(X(s)-X(t))^2+(Y(s)-Y(t))^2}\,ds \\
&\quad\quad - \int_0^\infty \frac{(X(s)+X(t))(u(s)Y'(s)-v(s)X'(s))-(Y(s)-Y(t))(u(s)X'(s)+v(s)Y'(s))}{(X(s)+X(t))^2+(Y(s)-Y(t))^2}\,ds \\
&\quad\quad + \int_0^\infty \frac{(X(s)-X(t))(v(s)X'(s)-u(s)Y'(s))+(Y(s)+Y(t))(u(s)X'(s)+v(s)Y'(s))}{(X(s)-X(t))^2+(Y(s)+Y(t))^2}\,ds \\
&\quad\quad + \int_0^\infty \frac{(X(s)+X(t))(v(s)X'(s)-u(s)Y'(s))+(Y(s)+Y(t))(u(s)X'(s)+v(s)Y'(s))}{(X(s)+X(t))^2+(Y(s)+Y(t))^2}\,ds. \qquad (10)
\end{aligned}
$$

On the free surface, the kinematic condition and Bernoulli equation yield

$$\big(u(t)-\omega Y(t)+\omega+1\big)Y'(t)=v(t)X'(t) \qquad (11)$$

and

$$\big(u(t)-\omega Y(t)+\omega+1\big)^2+v(t)^2-2\frac{\tau}{F^2}\frac{Y''(t)X'(t)-X''(t)Y'(t)}{(X'(t)^2+Y'(t)^2)^{3/2}}+\frac{2}{F^2}Y=1+\frac{2}{F^2}. \qquad (12)$$

Equation (12) expresses the fact that there is a jump in pressure across the free surface. This jump is proportional to the curvature of the free surface.

For given values of ω, τ and α we seek four functions u, v, X' and Y' satisfying (8)–(12). The parameter F is found as part of the solution.

2.2. Periodic waves

We now extend the formulation of subsection 2.1, to the case of a two-dimensional periodic wave of wavelength l. We use the dimensionless variables of subsection 2.1 and define ψ by (6). The precise definition of H and C will be given at the end of this subsection. We satisfy the kinematic boundary condition $v=0$ on the bottom by reflecting the flow into the bottom. Next we map the flow domain within $-l/2<x<l/2$ from the z-plane into the interior of an annular region of the ζ-plane by the transformation

$$\zeta=\mathrm{e}^{-\frac{2i\pi z}{l}}. \qquad (13)$$

Since $W(z)$ is periodic in x with period l, it gives rise to a single -valued analytic function of ζ. The boundaries of the annular region consist of the free surface and its image into the bottom. As in the case of solitary waves, we can apply the Cauchy integral formula to the function $W=u-iv$ but this time in the annular region in the ζ-plane. This yields

$$W(\zeta)=-\frac{1}{\pi i}\oint_{C_*}\frac{W(\mu)}{\mu-\zeta}\,d\mu, \qquad (14)$$

where ζ is on the free surface and C_* denotes here the boundaries of the annular region. The integral in (14) is a Cauchy-principal value.

We parameterize the free surface by $x=X(t),\ y=Y(t)$ where t is the arclength with $t=0$ at a crest of the wave. We denote by b, the value of t at $x=l$. Then

$$X'(t)^2+Y'(t)^2=1, \qquad (15)$$

$$X(0)=0, \qquad Y(0)=1+\alpha. \qquad (16)$$

Let

$$V_1 = u(s)X'(s) + v(s)Y'(s) \tag{17}$$

and

$$V_2 = u(s)Y'(s) - v(s)X'(s). \tag{18}$$

Then taking the real part of (14) yields after some algebra

$$\begin{aligned}
\frac{l}{2}u(t) \\
= \int_0^{\frac{b}{2}} & \frac{V_1(1 - \mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(s))}\cos(\frac{2\pi}{l}(X(t)+X(s)))) - V_2(\mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(s))}\sin(\frac{2\pi}{l}(X(t)+X(s))))}{1 + \mathrm{e}^{\frac{4\pi}{l}(Y(t)-Y(s))} - 2\mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(s))}\cos(\frac{2\pi}{l}(X(t)+X(s)))}\,ds \\
+ \int_0^{\frac{b}{2}} & \frac{V_1(1 - \mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(s))}\cos(\frac{2\pi}{l}(X(t)-X(s)))) + V_2(\mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(s))}\sin(\frac{2\pi}{l}(X(t)-X(s))))}{1 + \mathrm{e}^{\frac{4\pi}{l}(Y(t)-Y(s))} - 2\mathrm{e}^{\frac{2\pi}{l}(Y(t)-Y(w))}\cos(\frac{2\pi}{l}(X(t)-X(s)))}\,ds \\
- \int_0^{\frac{b}{2}} & \frac{V_1(1 - \mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\cos(\frac{2\pi}{?}(Y(t)+Y(s)))) - V_2(\mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\sin(\frac{2\pi}{l}(X(t)-X(s))))}{1 + \mathrm{e}^{\frac{4\pi}{l}(Y(t)+Y(s))} - 2\mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\cos(\frac{2\pi}{l}(X(t)-X(s)))}\,ds \\
- \int_0^{\frac{b}{2}} & \frac{V_1(1 - \mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\cos(\frac{2\pi}{l}(X(t)+X(s)))) + V_2(\mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\sin(\frac{2\pi}{l}(X(t)+X(s))))}{1 + \mathrm{e}^{\frac{4\pi}{l}(Y(t)+Y(s))} - 2\mathrm{e}^{\frac{2\pi}{l}(Y(t)+Y(s))}\cos(\frac{2\pi}{l}(X(t)+X(s)))}\,ds.
\end{aligned} \tag{19}$$

The condition of constant pressure on the free surface yields

$$\bigl(u(t) - \omega Y(t) + \omega + 1\bigr)^2 + v(t)^2 - 2\frac{\tau}{F^2}\frac{Y''(t)X'(t) - X''(t)Y'(t)}{(X'(t)^2 + Y'(t)^2)^{3/2}} + \frac{2}{F^2}Y(t) = B. \tag{20}$$

Here B is the Bernoulli constant and F is Froude number defined in (3).

Finally we define H and C uniquely by imposing the conditions

$$\int_0^b YX'\,ds = 2\int_0^{\frac{b}{2}} YX'\,ds = 0 \tag{21}$$

and

$$\int_0^b (uX' + vY')\,ds = 2\int_0^{\frac{b}{2}} (uX' + vY')\,ds = 0. \tag{22}$$

The condition (21) defines H as the average depth. Since the curl of the vector (u, v) is zero, an application of Stokes theorem shows that (22) is equivalent to the condition

$$\int_0^{l/2} u\,dx = 0.$$

where the integral is evaluated at a level $y =$ constant inside the flow domain.

For given values of ω, τ, l and α, we seek four functions u, v, X' and Y' satisfying (19)–(22). The values of F, B and b are found as part of the solution.

3. Analytical solution for linear periodic waves of small amplitude

In this section we derive an analytic solution for periodic waves of small amplitude. This linear solution will be useful in our discussion of generalized solitary waves. Since we will use different scalings, it is convenient to derive the solution in dimensional variables and to introduce appropriate scalings later. Thus we rewrite (6) as

$$\Psi = \psi - \frac{1}{2}\Omega y^2 + (\Omega H + C)y. \tag{23}$$

The exact nonlinear equations for the function ψ are

$$\nabla^2\psi = 0, \quad 0 < y < H + \zeta(x), \tag{24}$$

$$\psi_x = 0, \quad y = 0, \tag{25}$$

$$\psi_x = -(\psi_y - \Omega y + \Omega H + C)y_x, \quad y = H + \zeta(x), \tag{26}$$

$$\frac{1}{2}\left[\psi_x^2 + (\psi_y - \Omega y + \Omega H + C)^2\right] + gy - \frac{T}{\rho}\frac{y_{xx}}{(1 + y_x^2)^{3/2}} = B, \quad y = H + \zeta(x). \tag{27}$$

Here $y = H + \zeta(x)$ is the equation of the free surface and B is the Bernoulli constant.

We derive a linearized solution by assuming a small perturbation around the uniform shear flow $\psi = 0$, $\zeta = 0$. Thus we write

$$\psi = \varepsilon\psi_1(x, y) + \mathrm{O}(\varepsilon^2), \tag{28}$$

$$y = H + \varepsilon\zeta_1(x) + \mathrm{O}(\varepsilon^2), \tag{29}$$

$$C = C_0 + \varepsilon C_1 + \mathrm{O}(\varepsilon^2), \tag{30}$$

$$B = B_0 + \varepsilon B_1 + \mathrm{O}(\varepsilon^2). \tag{31}$$

Here $0 < \varepsilon \ll 1$ is a measure of the wave amplitude. Substituting (28)–(31) into (24)–(27), we obtain after some algebra

$$B_0 = \frac{1}{2}C_0^2 + gH$$

and the equations

$$\psi_{1xx} + \psi_{1yy} = 0, \quad 0 < y < H, \tag{32}$$

$$\psi_{1x}(x, 0) = 0, \quad y = 0, \tag{33}$$

$$\psi_{1x}(x, H) + C_0\zeta_{1x}(x) = 0, \quad y = H, \tag{34}$$

$$C_0C_1 + C_0\psi_{1y}(x, H) + g\zeta_1(x) - \Omega C_0\zeta_1(x) - \frac{T}{\rho}\zeta_{1xx}(x) = B_1, \quad y = H. \tag{35}$$

Using separation of variables, we write the solutions of (32), (33) as

$$\psi_1(x, y) = A_1 \sinh(ky)\cos(kx), \tag{36a}$$

$$\psi_1(x, y) = A_2 \mathrm{e}^{-kx}\sin(ky), \tag{36b}$$

where k is a constant.

Next we eliminate ζ_1 between (34) and (35). This yields

$$C_0\psi_{1yx} - g\frac{\psi_{1x}}{C_0} + \Omega\psi_{1x} + \frac{T}{\rho C_0}\psi_{1xxx} = 0 \quad \text{on } y = H. \tag{37}$$

Substituting (36) in (37) we obtain the relations

$$C_0^2 = \left(\frac{g - \Omega C_0}{k} + \frac{T}{\rho}k\right)\tanh(kH), \tag{38a}$$

$$C_0^2 = \left(\frac{g - \Omega C_0}{k} - \frac{T}{\rho}k\right)\tan(kH). \tag{38b}$$

Substituting (36) into (34) and integrating with respect to x give the equation for the free surface profiles

$$\zeta_1(x) = -\frac{A_1}{C_0}\sinh(kH)\cos(kx), \tag{39a}$$

$$\zeta_1(x) = -\frac{A_2}{C_0}\mathrm{e}^{-kx}\sin(kH). \tag{39b}$$

Therefore we have

$$y = H - \varepsilon\frac{A_1}{C_0}\sinh(kH)\cos(kx), \tag{40a}$$

$$y = H - \varepsilon\frac{A_2}{C_0}\mathrm{e}^{-kx}\sin(kH). \tag{40b}$$

The solution corresponding to (36a), (38a) and (39a) is a linear wave propagating at a constant phase velocity C_0 at the surface of a layer of fluid of depth H with constant vorticity Ω. The constant k is the wavenumber and the wavelength l is defined by $l = 2\pi/k$. The solution defined by (36b), (38b) and (39b) is a flow approaching the linear shear flow $\zeta_1 = 0$, $\psi_1 = 0$ as $x \to \infty$. It is not a wave, but we have included it because it describes the far field behavior of a solitary wave.

4. Analytical solution for solitary waves of small amplitude with $\tau \neq 1/3$

The linear theory of section 3 can be extended to the nonlinear regime by calculating terms of higher order in ε. This was done by Stokes for pure gravity irrotational waves and by Wilton [14] and others for gravity capillary irrotational waves (see Kang [15] for an extension to waves with constant vorticity). These expansions are not uniform as $H \to 0$ and different asymptotic solutions have to be derived when both the depth of the fluid and the amplitude of the wave are small. The appropriate expansions for irrotational gravity capillary waves are the Korteweg–de Vries equation and the fifth order Korteweg–de Vries equation. In this section, we derive the corresponding equations when the vorticity is constant.

We first introduce the following dimensionless variables

$$\bar{x} = \frac{x}{L}, \qquad \bar{y} = \frac{y}{H}, \qquad \eta = \frac{\zeta}{A}, \tag{41}$$

$$\bar{\psi} = \frac{H^2}{L^2AC}\psi. \tag{42}$$

In (41), (42), L is a lengthscale of the wave in the x-direction and A is a measure of the amplitude.

Substituting (41) and (42) in (24)–(27) with $\frac{1}{2}C^2 + gH$ instead of B we obtain (after dropping the bars) the following exact nonlinear equations for ψ

$$\beta\psi_{xx} + \psi_{yy} = 0, \quad 0 < y < 1 + \alpha\eta, \tag{43}$$

$$\psi_x = 0 \quad \text{on } y = 0, \tag{44}$$

$$\psi_x = -(\alpha\psi_y - \alpha\beta\omega\eta + \beta)\eta_x \quad \text{on } y = 1 + \alpha\eta, \tag{45}$$

$$\frac{1}{2}\frac{\alpha}{\beta}\psi_x^2 + \frac{1}{2}\frac{\alpha}{\beta^2}\left(\psi_y - \beta\omega\eta + \frac{\beta}{\alpha}\right)^2 + \frac{1}{F^2}\eta - \beta\tau^*\frac{\eta_{xx}}{(1+\alpha^2\beta\eta_x^2)^{3/2}} = \frac{1}{2\alpha} \quad \text{on } y = 1 + \alpha\eta. \tag{46}$$

In (43)–(46), α and β are the dimensionless parameters

$$\alpha = \frac{A}{H}, \qquad \beta = \frac{H^2}{L^2} \tag{47}$$

and

$$\tau^* = \frac{T}{\rho C^2 H} = \frac{\tau}{F^2}. \tag{48}$$

The Korteweg–de Vries equation for irrotational flow describes waves of small amplitude (A small) in shallow water (H small). The precise limit is obtained by letting

$$\alpha = \varepsilon \tag{49}$$

and

$$\beta = \varepsilon \tag{50}$$

and taking the limit as $\varepsilon \to 0$. The approach is similar to the one presented in Whitham [16] for gravity waves without vorticity and by Hunter and Vanden-Broeck [2] for gravity capillary waves without vorticity. Therefore only the main steps of the calculations will be presented. Full details can be found in Kang [15]. Expanding η, ψ and F as

$$\begin{aligned} \eta &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \\ \psi &= \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \cdots, \\ F &= F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots \end{aligned} \tag{51}$$

and substituting (51) into (43)–(46), we obtain, after some algebra, the equations

$$F_0^2 = \frac{1}{\omega + 1}, \tag{52}$$

$$\psi_1 = -\eta_0 y, \tag{53}$$

$$\psi_2 = \frac{1}{6}\eta_{0xx}y^3, \tag{54}$$

$$\eta_1 = \frac{2+\omega}{2}\eta_0^2 - \frac{1}{6}\eta_{0xx}, \tag{55}$$

$$\frac{2F_1}{F_0^3}\eta_{0x} - (\omega^2 + 3\omega + 3)\eta_0\eta_{0x} - \left(\frac{1}{3} - \tau^*\right)\eta_{0xxx} = 0. \tag{56}$$

An alternative derivation of (56) based on the work of Benjamin [7] for gravity waves with vorticity can be found in Kang [15].

Equation (56) admits the solitary wave solution

$$\eta_0 = a\,\mathrm{sech}^2\frac{x}{2b}, \quad a = \frac{6F_1}{F_0^3(\omega^2+3\omega+3)}, \; b = \left[\frac{1-3\tau^*}{(\omega^2+3\omega+3)a}\right]^{1/2}. \tag{57}$$

When $\tau^* < 1/3$, (57) is an elevation wave with Froude number greater than $1/\sqrt{1+\omega}$ and when $\tau^* > 1/3$, it is a depression wave with Froude number less than $1/\sqrt{1+\omega}$. Since b in (57) vanishes when $\tau^* = 1/3$, equation (56) is invalid in the neighbourhood of $\tau^* = 1/3$.

We shall now follow the work of Hunter and Vanden-Broeck [2] to derive an equation analogous to the equation (56) which is valid in a neighbourhood of $\tau^* = 1/3$. The coefficient of the dispersive term η_{0xxx} in (56) vanishes at $\tau^* = 1/3$. To obtain a balance between the dispersive and nonlinear terms near $\tau^* = 1/3$ we take

$$\alpha = \varepsilon^2, \qquad \beta = \varepsilon \tag{58}$$

in (43)–(46).

Then we expand η, ψ, B and τ^* as

$$\begin{aligned}
\psi &= \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \cdots, \\
\eta &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \\
\tau^* &= \frac{1}{3} + \varepsilon\tau_1^* + \varepsilon^2\tau_2^* + \cdots, \\
F &= F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots, \\
\frac{1}{F^2} &= \frac{1}{F_0^2} - \varepsilon\frac{2F_1}{F_0^3} + \varepsilon^2\left(\frac{3F_1^2}{F_0^4} - \frac{2F_2}{F_0^3}\right) + \cdots.
\end{aligned} \tag{59}$$

Substituting (58) and (59) into (43)–(46), we obtain, after some algebra,

$$\frac{1}{F_0^2} = 1+\omega, \qquad F_1 = 0, \tag{60}$$

$$\eta_0 = -\psi_{1y}, \tag{61}$$

$$\frac{2F_2}{F_0^3}\eta_{0x} - \left(\omega^2+3\omega+3\right)\eta_0\eta_{0x} + \tau_1^*\eta_{0xxx} - \frac{1}{45}\eta_{0xxxxx} = 0. \tag{62}$$

The equation (62) with $\omega = 0$, has been studied by many authors (see for example Kawahara [17]; Hunter and Vanden-Broeck [2]; Champneys and Toland [18], and Calvo, Yang and Akylas [19]). Our analysis shows that the effect of constant vorticity is simply to modify the coefficient of $\eta_0\eta_{0x}$ in (62).

5. Discussion of the analytical results

In this section we discuss some of the implications of the analytical results of sections 3 and 4. In particular, we identify ranges of parameters for which we can expect generalized solitary waves. First we note that (38a) reduces to the usual dispersion relation for linear gravity capillary irrotational waves if we replace $g - \Omega C_0$

by g. This suggests the definition of a modified Froude number $\bar{F}$ and a modified Bond number $\bar{\tau}$ by the relations

$$\bar{F}^2 = \frac{C_0^2}{(g - \Omega C_0)H}, \tag{63}$$

$$\bar{\tau} = \frac{T}{\rho(g - \Omega C_0)H^2}. \tag{64}$$

In terms of $\bar{F}$ and $\bar{\tau}$ we rewrite (38a) as

$$\bar{F}^2 = \left(\frac{1}{kH} + \bar{\tau}kH\right)\tanh kH. \tag{65}$$

Relation (65) defines $\bar{F}$ as a function of kH for each given value of $\bar{\tau}$.

For $kH \ll 1$, (65) gives

$$\bar{F}^2 = 1 + \left(\bar{\tau} - \frac{1}{3}\right)(kH)^2 + \mathrm{O}\left[(kH)^4\right]. \tag{66}$$

For $kH \gg 1$, (65) gives

$$\bar{F}^2 \approx \bar{\tau}kH. \tag{67}$$

For $\bar{\tau} < 1/3$, $\bar{F}^2$ has a minimum for some value of kH. This can be checked by noticing that $\bar{F}$ is an increasing function of kH for kH large and a decreasing function of kH for kH small (see (66) and (67)). For $\bar{\tau} > 1/3$, $\bar{F}^2$ is a monotonically increasing function of kH. In particular $\bar{F} > 1$ for all kH.

We now use these results together with those of section 4 to discuss the possible existence of generalized solitary waves. In section 4, we found solitary waves characterized by

$$F^2 = \frac{1}{1+\omega} + \mathrm{O}\left(a^2\right) \tag{68}$$

as $a \to 0$, with the properties

$$F > \frac{1}{\sqrt{1+\omega}} \quad \text{for } \tau^* < \frac{1}{3}, \tag{69}$$

$$F < \frac{1}{\sqrt{1+\omega}} \quad \text{for } \tau^* > \frac{1}{3}. \tag{70}$$

Relations (2), (3) and (63) show that (69) implies

$$\bar{F} > 1$$

and that (70) implies

$$\bar{F} < 1.$$

Furthermore (1), (64) and (68) give

$$\bar{\tau} = \tau^* + \mathrm{O}\left(a^2\right) \quad \text{as } a \to 0. \tag{71}$$

Therefore, for a sufficiently small, we can rewrite (69) and (70) as

$$\bar{F} > 1 \quad \text{for } \bar{\tau} < \frac{1}{3}, \tag{72}$$

$$\bar{F} < 1 \quad \text{for } \bar{\tau} > \frac{1}{3}. \tag{73}$$

As noticed earlier, linear periodic waves for $\bar{\tau} > 1/3$ are characterized by $\bar{F} > 1$. Therefore (73) shows that there are no linear periodic waves travelling at the same speed as the solitary wave of section 4 for $\bar{\tau} > 1/3$. On the other hand, for linear periodic waves with $\bar{\tau} < 1/3$, $\bar{F}$ first decreases from 1 as kH increases and then increases to infinity as $kH \to \infty$. Therefore for a small, (72) shows that there is a linear periodic wave travelling at the same speed as the solitary wave of section 4 for $\bar{\tau} < 1/3$. By analogy with the theory of irrotational gravity capillary waves, this strongly suggests that the free surface of the solitary wave may contain a train of periodic waves in the far field. In other words the solitary wave of section 4 for $\tau^* < 1/3$ might be incorrect because it predicts a flat free surface in the far field instead of an oscillatory tail. This is confirmed by the numerical calculations presented in section 6. We note that the amplitude of the oscillatory tail is exponentially small in a as $a \to 0$. Finally let us mention that the considerations of this section are restricted to waves of small amplitude. On the other hand, the numerical computations of section 6 are fully nonlinear and do not preassume that the amplitude of the wave is small.

6. Discussion of the numerical results

Numerical schemes based on the integro-differential equations of section 2, can be used to solve the problem in the fully nonlinear case. Such schemes were presented in Vanden-Broeck [10,11] for gravity solitary waves with constant vorticity and in Vanden-Broeck [12] for gravity periodic waves with constant vorticity. Here we adapt these schemes to include the effect of surface tension. The essential modification is to evaluate the higher order derivatives in the curvature in (12) and (20) by finite differences. In the following discussion, we refer to the scheme for solitary wave as the scheme A and to the scheme for periodic waves as the scheme B.

6.1. Depression solitary waves

The numerical scheme A was used to compute depression solitary waves for given values of τ^*, U(0) and ω. Here $U(0)$ is the velocity on the free surface at $x = 0$. The parameter $U(0)$ can be viewed as a measure of the size of the wave ($U(0)$ close to one corresponds to a wave of small amplitude). We choose ω greater than -1, so that $F_0^2 > 0$ in (52). The asymptotic solution (57) can be rewritten as

$$y = 1 + A\left[\operatorname{sech}^2\left[\frac{3A}{4(1-3\tau^*)}\right]^{1/2} x\right], \tag{74}$$

$$F^2 = \frac{1}{1+\omega} + \frac{\omega^2 + 3\omega + 3}{3(1+\omega)^{3/2}} A. \tag{75}$$

This asymptotic solution was used as the initial guess in the Newton iterations involved in the numerical scheme. Once a solution was obtained, it was used as the initial guess for the next calculation with a slightly different value of τ^*, U(0) or ω. Most calculations were performed with 200 mesh points on the portion $x > 0$ of the free surface profile.

We first compared the numerical results with the approximation (74), (75) for various values of U(0) and $-0.5 < \omega < 0.5$ for $\tau^* = 0.4$ and $\tau^* = 0.7$. We found that the numerical values of F agree with (75) as $U(0) \to 1$. This constitutes a check on the numerical scheme. As $U(0)$ increases, the approximation (74)

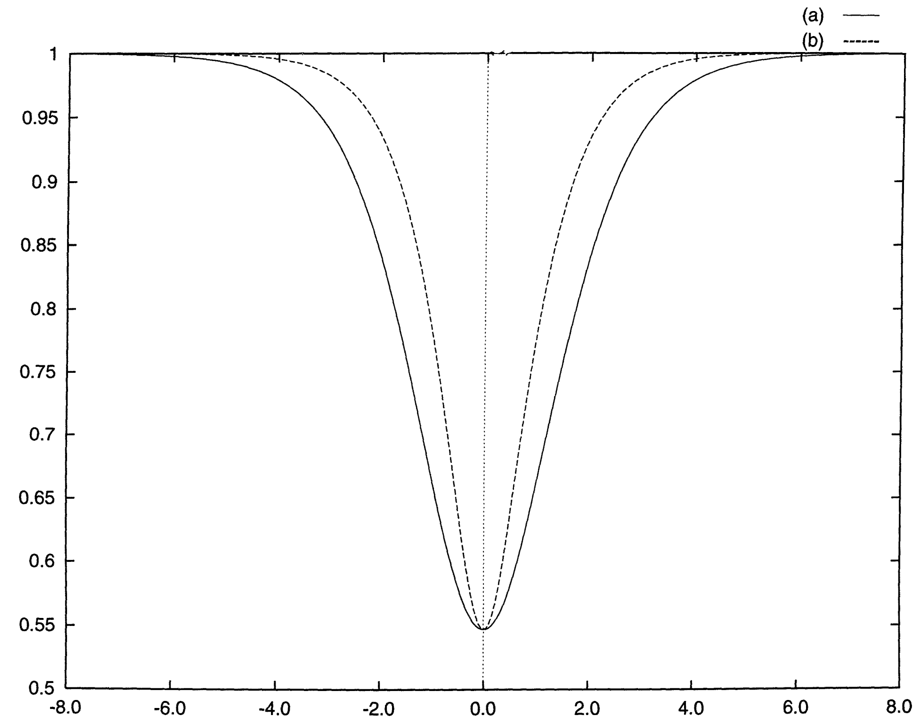

Figure 1. (1) Free-surface profiles of depression solitary waves for $\tau^* = 0.7$ and $\omega = -0.3$. The curve (a) corresponds to the KdV approximation (74) in which A is equal to the amplitude of curve (b). The curve (b) is the numerical solution for $U(0) = 2.0$. Here the unit length is H.

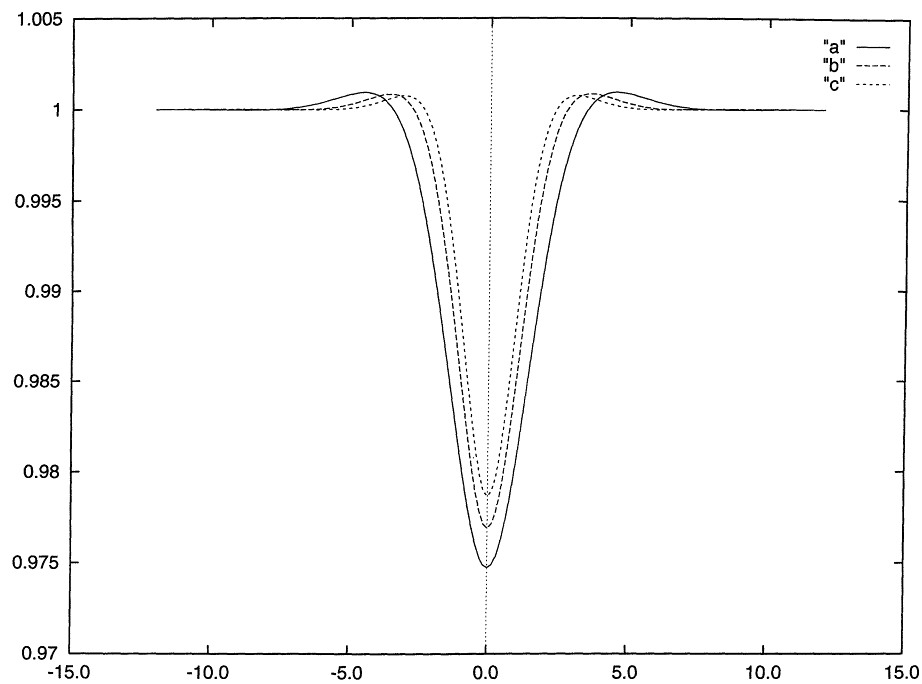

Figure 2. Computed free surface profile of a depression solitary wave for $U(0) = 1.03$ and $\tau^* = 0.33$ with (a): $\omega = 0$, (b): $\omega = 1.0$ and (c): $\omega = 2.0$.

becomes poor. This is illustrated in *figure 1* where we compare the numerical free surface profile with the approximation (74) for $U(0)=2$, $\omega=-0.3$ and $\tau^*=0.7$. *Figure 2* shows computed free surface profiles for $U(0)=1.03$, $\tau^*=0.33$ and various values of ω. As ω decreases the distance from the through to the bottom of the channel decreases.

In *figure 3*, we present solitary waves with decaying oscillatory tails. Such waves have been studied intensively for irrotational flows (see for Hunter and Vanden-Broeck [2], Vanden-Broeck and Dias [20], and Dias, Menasce and Vanden-Broeck [21]) for numerical studies). The present computations show that such waves exist in the presence of constant vorticity.

6.2. Periodic waves and elevation solitary waves

In this subsection, we present results for generalized solitary waves. These waves form branches of solutions distinct from the branches of solutions with decaying tails. The discussion of section 5 suggests that generalized solitary waves should exist for $0<\bar{\tau}<1/3$. Since generalized solitary waves are characterized by nondecaying oscillatory tails in the far field, Y does not approach 1 in the far field. Following Hunter and Vanden-Broeck [2] and Vanden-Broeck [6], we overcome the difficulties associated with the choice of the appropriate boundary conditions in the far field, by approximating the generalized solitary waves by periodic waves with a very long wavelength l. As $l\to\infty$, such periodic waves approach generalized solitary waves. We used the numerical scheme B to compute periodic waves of large wavelength l for various values of $0<\tau^*<1/3$, $U(0)$, ω and l. Here $U(0)$ is the velocity at the main crest of the wave. The advantage of scheme B over scheme A is that the periodicity of the flow enables us to calculate solutions without worrying about the truncation in the far field. It was found that for each value of ω and $0<\tau^*<1/3$, there are many different families of solutions. This non-uniqueness agrees with the known results for irrotational waves (see, for example, Schwartz and Vanden-Broeck [22]). For given values of τ^* and ω, there is a two-parameter family of solutions. These parameters can be chosen as $U(0)$ and l. Since we want to approximate solitary waves, we restrict our attention to large values of l (around 100). Most of the computations were performed with 400 mesh points in the portion $x>0$ of the free surface profile.

In the computations, we used an inverse approach in which we fix $\tau^*,\ U(0),\ F$ and ω and find l as part of the solution. Some free surface profiles are shown in *figure 4*. These results extend the irrotational findings of Hunter and Vanden-Broeck [2] and Vanden-Broeck [6]. We note that there are particular values of the parameters for which the amplitude of the ripples is essentially zero. Such waves were calculated for $\omega=0$ by Vanden-Broeck [6].

6.3. Steep solitary waves

The numerical results presented in the two previous subsections are restricted to waves of small or moderate amplitudes. In this subsection, we present one result for large waves.

Vanden-Broeck [10,11] computed gravity solitary waves with constant vorticity. He found that there are branches of solutions which approach a uniform shear flow as their amplitude tends to zero in agreement with the previous findings of Benjamin [7], Teles da Silva and Peregrine [8] and Pullin and Grimshaw [9]. Furthermore he obtained additional branches of solutions which do not bifurcate from a uniform shear flow and exist only for sufficiently large values of the amplitude. These branches of solutions have limiting configurations with circular regions of fluid in rigid body rotation at their crests.

We have used the scheme A to generalize the findings of Vanden-Broeck [10,11] by including the effect of surface tension. The results were found to be qualitatively similar. A new feature is the existence of solutions

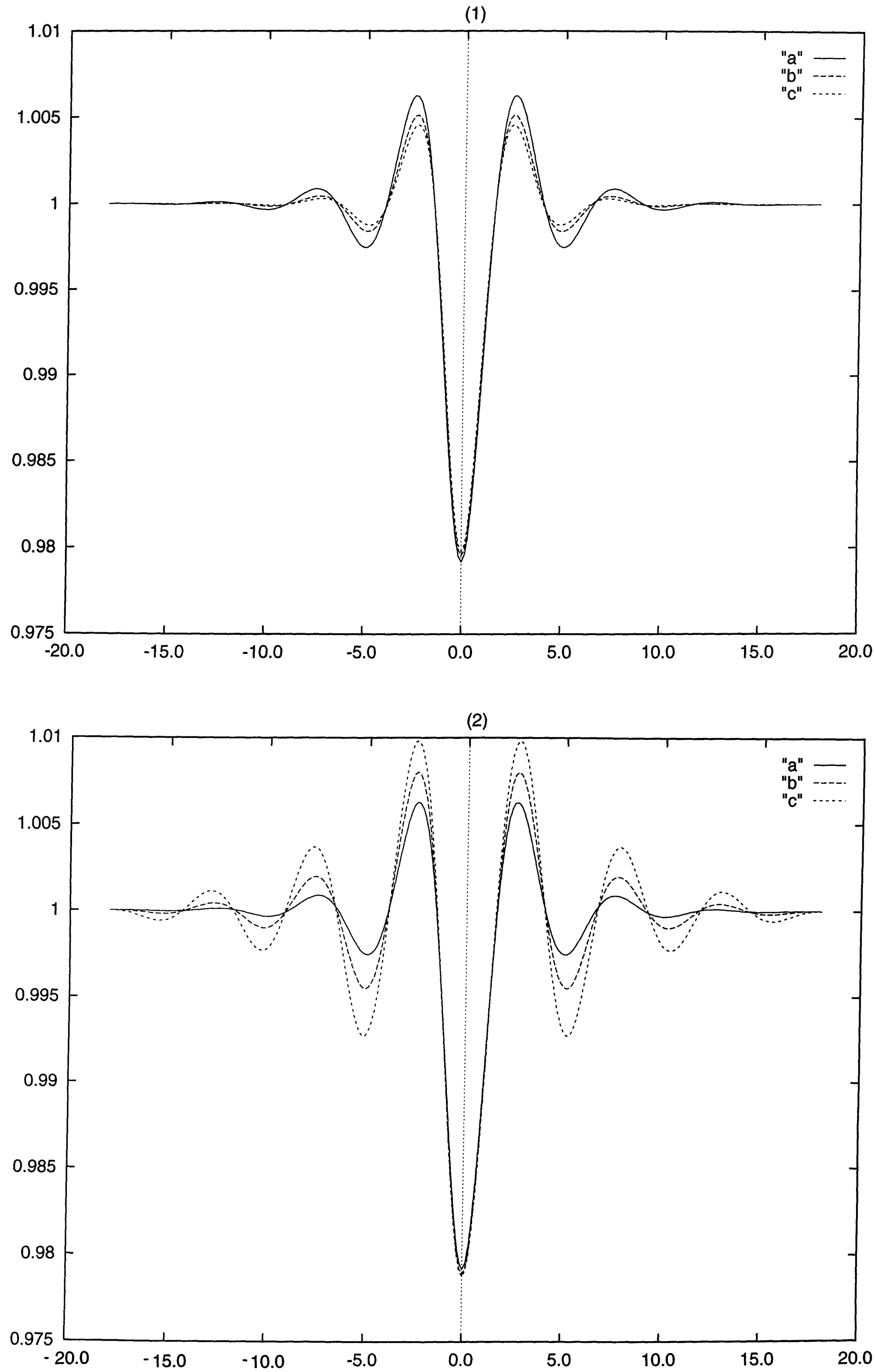

Figure 3. (1) Computed free surface profile of a depression solitary wave for $U(0) = 1.03$ and $\tau^* = 0.28$ with (a): $\omega = 0$, (b): $\omega = 0.3$ and (c): $\omega = 0.5$. (2) Same as (1) with (a): $\omega = 0$, (b): $\omega = -0.3$ and (c): $\omega = -0.5$.

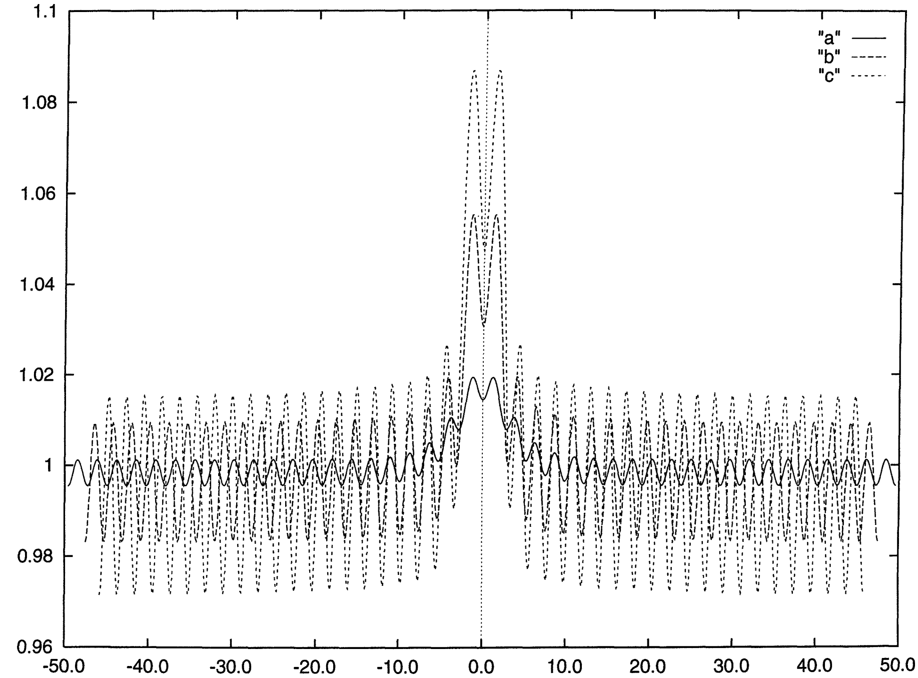

Figure 4. (1) Computed free surface profile of a long periodic wave with $\tau^* = 0.24$, $U(0) = 0.99$ and $F = 1.00255$ for (a): $\omega = 0.01$, (b): $\omega = 0.05$ and (c): $\omega = 0.09$.

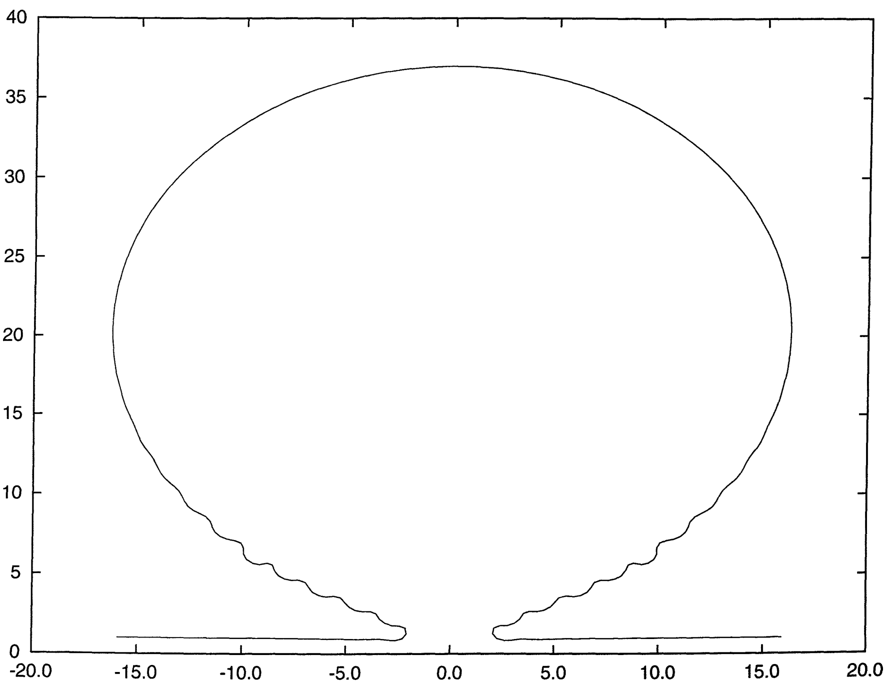

Figure 5. Computed free surface profile of wave with $A = 36$, $\omega = -0.11$ and $\tau = 253$.

with a train of ripples on the central part of the profile (see *figure 5*). This profile is to be contrasted with the profiles without surface tension in figure 6 of Vanden-Broeck [10] (see also those in Vanden-Broeck [11]) in which such ripples do not occur. It is interesting to note the similarity with the profiles obtained by Debiane and Kharif [23].

7. Conclusions

We have generalized the theory of gravity capillary water waves by including the effect of vorticity. The vorticity was assumed to be constant within the fluid. Numerical and analytical solutions were presented for periodic and solitary waves. It was shown that there are both solitary waves with decaying oscillatory tails and solitary waves with oscillatory tails of constant amplitude. The latter are referred to as generalized solitary waves. These branches of solutions bifurcate from a uniform shear flow. We have shown that there are in addition branches which exist only if the amplitude of the waves is sufficiently large.

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